Is it possible that these expansions are identical except for the sign of the odd powers of δ^4?

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Continuous Spectrum and Characteristic Modes of the Slot Line in Free Space

Marat Davidovitz

***Abstract*—The continuous spectrum of the slot in an infinite ground plane is expressed in terms of the Mathieu functions. The so-called characteristic values and field modes of the slot are stated explicitly.**

I. CONTINUOUS SPECTRUM REPRESENTATION

In a recent paper [1] a general eigenspectrum construction method for open waveguides was presented. As an illustration, the case of the slotted screen was considered. Extensive analysis of the same problem was also undertaken in [2] and subsequent publications by the same authors. The purpose of this note is to state in closed form the eigenmodes, the characteristic slot-field modes and values for the aforementioned geometry. These may be used to verify numerical solutions, as basis functions for more complex, nonseparable geometries, or to investigate slot-line discontinuities.

Consider an infinite slot of width a in a perfectly conducting, zero-thickness screen. Let the center line of the slot define the z-axis, and the x-axis lie in the plane of the screen. The fields in this structure can be represented in terms of a complete, orthonormal set of z-guided eigenfunctions, each satisfying appropriate boundary conditions on the screen. The transverse-to-z cross section of the structure is unbounded and homogeneously filled; the eigenspectrum is continuous and allows decomposition into TE_z and TM_z components [3]

For a complete description of the notation used here for the elliptic cylindrical coordinates and the Mathieu functions the reader is referred to [4].

A. TM_z Eigenmodes

The transverse electric field of a TM_z eigenmode can be represented as the gradient of a scalar function $\Phi_m(h, \cosh\mu, \cos\theta)$, where $h = \frac{1}{2}k_t a$, $0 \leq k_t \leq \infty$ is the continuous spectral variable, m is the discrete index associated with the angular solutions and μ, θ denote, respectively, the radial and angular elliptic coordinates.

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Following the approach employed in [5], [6] for the case of a ridged elliptical waveguide, eigenmodes with even or odd symmetry with respect to the plane of the screen are distinguished.

The even solutions are given by

$$
\begin{aligned}
\Phi^e_m(h, \cosh\mu, \cos\theta) &= Po_m(h, \cosh\mu) So_m(h, \cos\theta) \\
Po_m(h, \cosh\mu) &= No'_m(h, 1) Jo_m(h, \cosh\mu) \\
&\quad - Jo'_m(h, 1) No_m(h, \cosh\mu)
\end{aligned}
$$

and the odd solutions by

$$
\begin{aligned}
\Phi^o_m(h, \cosh\mu, \cos\theta) &= Qo_m(h, \cosh\mu) So_m(h, \cos\theta) \\
Qo_m(h, \cosh\mu) &= \frac{Jo_m(h, \cosh\mu)}{Jo'_m(h, 1)}
\end{aligned}
$$

where $So_m(h, \cos\theta)$ is the m-th odd angular Mathieu function, $Jo_m(h, \cosh\mu)$ and $No_m(h, \cosh\mu)$ are, respectively, the radial Mathieu functions of the first and second kind associated with the odd angular solution, and prime denotes differentiation with respect to μ. The boundary conditions on the screen are satisfied by virtue of the fact that $So_m(h, \cos\theta) = 0$ at $\theta = 0, \pi$. Among other properties of the two solutions are the following facts

$$
\begin{aligned}
&Po_m(h, 1) = 1 \quad Po'_m(h, 1) = 0 \\
&Qo_m(h, 1) = 0 \quad Qo'_m(h, 1) = 1 \\
&\mathcal{W}(Po_m(h, \cosh\mu), Qo_m(h, \cosh\mu)) = 1
\end{aligned}
$$

where $\mathcal{W}$ denotes the Wronskian.

B. TE_z Eigenmodes

The transverse magnetic field of a TE_z eigenmode can be represented as the gradient of a scalar function $\Psi_m(h, \cosh\mu, \cos\theta)$, where the previously introduced notation is applicable.

The even solutions are given by

$$
\begin{aligned}
\Psi^e_m(h, \cosh\mu, \cos\theta) &= Pe_m(h, \cosh\mu) Se_m(h, \cos\theta) \\
Pe_m(h, \cosh\mu) &= Je_m(h, 1) Ne_m(h, \cosh\mu) \\
&\quad - Ne_m(h, 1) Je_m(h, \cosh\mu)
\end{aligned}
$$

and the odd solutions by

$$
\begin{aligned}
\Psi^o_m(h, \cosh\mu, \cos\theta) &= Qe_m(h, \cosh\mu) Se_m(h, \cos\theta) \\
Qe_m(h, \cosh\mu) &= \frac{Je_m(h, \cosh\mu)}{Je_m(h, 1)}
\end{aligned}
$$

where $Se_m(h, \cos\theta)$ is the mth even angular Mathieu function, $Je_m(h, \cosh\mu)$ and $Ne_m(h, \cosh\mu)$ are, respectively, the radial Mathieu functions of the first and second kind associated with the even angular solution, and prime denotes differentiation with respect to μ. The boundary conditions on the screen are satisfied because $\frac{\partial Se_m(h, \cos\theta)}{\partial\theta} = 0$ at $\theta = 0, \pi$. Additional properties of the two solutions include

$$
\begin{aligned}
&Pe_m(h, 1) = 0 \quad Pe'_m(h, 1) = 1 \\
&Qe_m(h, 1) = 1 \quad Qe'_m(h, 1) = 0 \\
&\mathcal{W}(Pe_m(h, \cosh\mu), Qe_m(h, \cosh\mu)) = 1.
\end{aligned}
$$

C. Normalization Constants

In order to expand the fields of sources in the presence of the slotted screen, the eigenmodes must be normalized. This can be accomplished in a number of ways [3]. Only the final results are stated.

Let the transverse electric field of a TM_z mode corresponding to the mth radial solution and spectral number k_t be denoted by

$$\mathbf{E}_{\rm TM}^{e,o}(m,h) = \nabla_t \Phi_m^{e,o}(h, \cosh\mu, \cos\theta)$$

and the electric field of a TE_z mode corresponding to the mth radial solution and spectral number k_t by

$$\mathbf{E}_{\rm TE}^{e,o}(m,h) = \hat{\mathbf{z}} \times \nabla_t \Psi_m^{e,o}(h, \cosh\mu, \cos\theta).$$

Upon defining the inner product over the transverse cross-section S as follows

$$\langle \mathbf{A}, \mathbf{B} \rangle = \iint_S \mathbf{A} \cdot \mathbf{B} \quad ds$$

it can be shown that the following equations are true

$$\begin{aligned}
&\langle \mathbf{E}_{\rm TM}^e(m,h_1), \mathbf{E}_{\rm TM}^e(n,h_2)\rangle \\
&= \frac{\pi}{2} k_{t1} M_m^o(h_1)[Jo_m'(h_1,1)^2 + No_m'(h_1,1)^2] \\
&\quad \times \delta(k_{t1} - k_{t2})\delta_{mn} \\
&\langle \mathbf{E}_{\rm TM}^o(m,h_1), \mathbf{E}_{\rm TM}^o(n,h_2)\rangle \\
&= \frac{\pi}{2} k_{t1} M_m^o(h_1)[Jo_m'(h_1,1)]^{-2}\delta(k_{t1} - k_{t2})\delta_{mn} \\
&\langle \mathbf{E}_{\rm TE}^e(m,h_1), \mathbf{E}_{\rm TE}^e(n,h_2)\rangle \\
&= \frac{\pi}{2} k_{t1} M_m^e(h_1)[Je_m(h_1,1)^2 + Ne_m(h_1,1)^2] \\
&\quad \times \delta(k_{t1} - k_{t2})\delta_{mn} \\
&\langle \mathbf{E}_{\rm TE}^o(m,h_1), \mathbf{E}_{\rm TE}^o(n,h_2)\rangle \\
&= \frac{\pi}{2} k_{t1} M_m^e(h_1)[Je_m(h_1,1)]^{-2}\delta(k_{t1} - k_{t2})\delta_{mn}
\end{aligned}$$

where $\delta(\cdot)$ is the delta function, δ_{mn} is the Kronecker symbol, and normalization constants $M_m^{e,o}$ are defined in [4].

D. Characteristic Modes for the Slot

The characteristic slot-field modes defined in [2] can be shown to correspond to the tangential field distributions of the eigenmode solutions stated above, evaluated at $\mu = 0$. For example, the characteristic TE_z aperture modes for a given value of k_t are obtained as

$$\mathbf{E}_{\rm TE}^e(m,h,1,x) = \hat{\mathbf{x}} \frac{Se_m(h, 2x/a)}{\sqrt{(\frac{a}{2})^2 - x^2}}; \quad -\frac{a}{2} \le x \le \frac{a}{2}$$

where the substitution $\frac{1}{2}a\cos\theta = x$ was made. Numerical evaluation of $Se_m(h, 2x/a)$ can be carried out using a number of software packages, e.g., [7].

The characteristic values, denoted by $b_m(k_t)$ in [2] ($\lambda_m(k_t)$ in [1]) can be explicitly written as follows

$$b_m(k_t) = \frac{Ne_m(h,1)}{Je_m(h,1)}.$$

The approximate $b_m(k_t)$ values derived in [2] represent the first terms in the series expansion of the preceding equation for small values of h.

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Electrostatic Potential Through a Circular Aperture in a Thick Conducting Plane

Jung H. Lee and Hyo J. Eom

***Abstract*—The electrostatic potential through a circular aperture in a thick conducting plane is examined. The Hankel transform is applied to express the scattered potential in the spectral domain and the boundary conditions are enforced to obtain simultaneous equations for the transmitted potential inside the thick conducting plane. The simultaneous equations are solved to represent the transmitted and scattered potentials in series forms. Numerical computations are performed to illustrate the behavior of polarizability in terms of the aperture size. The numerical comparisons to other available data show excellent agreement. The presented series solution is fast convergent so that it is very efficient for numerical computation.**

I. Introduction

Electrostatic potential through a circular aperture in a thin conducting plane has been of considerable interest in the area of microwaves [1]–[3]. The potential penetration through a circular aperture in a thick conducting plane has been studied with the variational technique [4]. Although the solution in [4] fairly well agrees with the measurement data, it is also of interest to obtain another rigorous exact solution. The motivation of the present study is to develop such a solution by using the Hankel transform and the mode-matching used in [5]. The solution presented in this paper is in simple convergent series so that it is not only exact but also computationally very efficient. The organization of the paper is as follows: In the next section, we represent the scattered potential in the spectral domain and perform the numerical calculations. A brief summary is given in Conclusion.

II. Potential Representations and Boundary Conditions

In region (I) ($z > 0$), an incident potential Φ^i impinges on a circular aperture (radius: a, depth: d) in a thick conducting plane at zero potential (see Fig. 1). Regions (II) ($-d < z < 0, r < a$) and (III) ($z < -d$) denote the circular aperture and the lossless half-space, respectively. In region (I) the total potential consists of the

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